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LETTER TO THE EDITOR

# Berry's phase for coherent states

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**Abstract.** We calculate Berry's phase and the Hannay angle for some physically interesting coherent states with the parameters characterising the coherent states taken to be slowly varying. Interestingly, we find that the harmonic oscillator coherent states provide an example for which, although the Hannay angle is zero, Berry's phase is non-zero.

Phases in quantum mechanics often give rise to interesting observable physical phenomena. One such phase factor which has attracted considerable interest in recent years is Berry's phase (Berry 1984). This arises in the context of Hamiltonians $H(\boldsymbol{R}(t))$ which depend on slowly varying external parameters $\boldsymbol{R}(t)$. Berry has shown that, in the adiabatic approximation, the solution of the time-dependent Schrödinger equation, initially chosen to be a non-degenerate eigenstate of the instantaneous Hamiltonian

$$H(R)|n, R\rangle = E_n(R)|n, R\rangle \tag{1}$$

acquires, in addition to the usual 'dynamical phase factor', a geometrical phase factor given by

$$\gamma_n(c) = \mathrm{i}\int_0^T \mathrm{d}t\langle n, R|\mathrm{d}/\mathrm{d}t|n, R\rangle \tag{2}$$

as the parameters are slowly varied along a closed curve $c$ in the parameter space in time $T$. The expression (2) for $\gamma_n(c)$ may alternatively be written as

$$\gamma_n(c) = \mathrm{i}\int_c \mathrm{d}\boldsymbol{R}\langle n, \boldsymbol{R}|\nabla_R|n, \boldsymbol{R}\rangle. \tag{3}$$

A classical analogue of Berry's phase, called the Hannay angle, was later discovered by Berry himself and Hannay (Berry 1985, Hannay 1985). For a classical Hamiltonian $H(p, \boldsymbol{q}, \boldsymbol{R}(t))$ depending on slowly varying external parameters they showed that, in the adiabatic approximation, the angle variable $w$ conjugate to the action variable I acquires a shift given by

$$\Delta w(I, c) = -\int_c \mathrm{d}R\frac{\partial}{\partial I}\frac{1}{2\pi}\oint \mathrm{d}w\, p(w, I, R)\nabla_R\boldsymbol{q}(w, I, \boldsymbol{R}) \tag{4}$$

as one goes round a closed circuit $c$ in the parameter space. They further show that semiclassically Berry's phase $\gamma_n(c)$ and the Hanny angle $\Delta w(I, c)$ are related to each other by the following equation:

$$\Delta w(I, c) = -\hbar \frac{\partial}{\partial I} \gamma_n(c) = -\frac{\partial \gamma_n(c)}{\partial n}. \tag{5}$$

In the literature there exist several works directed towards

($a$) providing a mathematical interpretation to Berry's phase (Simon 1983);

($b$) generalising Berry's result to the degenerate case (Wilczek and Zee 1984);

($c$) establishing mathematical conditions for the existence of a non-trivial Berry phase (Kiritsis 1986);

($d$) relating Berry's phase to Wess-Zumino terms and to anomalies in field theories (Aitchison 1986, Sonoda 1986);

($e$) experimentally verifying the existence of Berry's phase (Tomita and Chiao 1986).

In this letter we study Hamiltonians which admit (i) harmonic oscillator coherent states, (ii) spin coherent states and (iii) squeezed coherent states as their eigenstates. We calculate Berry's phase and the Hannay angle for these states taking the two parameters characterising them as the slowly varying parameters.

First we consider harmonic oscillator coherent states.

*(a) Berry's Phase.* Consider the Hamiltonian for a displaced harmonic oscillator:

$$H = \hbar\omega[(a^+ - \alpha^*)(a - \alpha) + \tfrac{1}{2}] \tag{6a}$$

$$= D(\alpha)[\hbar\omega(a^+ a + \tfrac{1}{2})]D^+(\alpha) \tag{6b}$$

where

$$D(\alpha) = \exp(\alpha a^+ - \alpha^* a) \tag{7}$$

and $\alpha = X_1 + \mathrm{i}X_2$. We take $X_1$ and $X_2$ to be slowly varying parameters. As is well known the eigenstates of this Hamiltonian are the coherent states

$$|n, \alpha\rangle = D(\alpha)|n\rangle \tag{8}$$

where $|n\rangle$ are the usual harmonic oscillator states.

Berry's phase is given by

$$\gamma_n(c) = \int_c \mathrm{d}X \langle n, \alpha|\nabla_X|n, \alpha\rangle \tag{9}$$

$$= \int_c \mathrm{d}X \langle n|D^+(\alpha)\nabla_X D(\alpha)|n\alpha\rangle. \tag{10}$$

Using

$$D^+(\alpha)\partial D(\alpha)/\partial X_1 = -\mathrm{i}X_2 + (a^+ - a) \tag{11a}$$

$$D^+(\alpha)\partial D(\alpha)/\partial X_2 = \mathrm{i}X_1 + \mathrm{i}(a^+ + a) \tag{11b}$$

which follow from

$$D(\alpha) = \exp(-\tfrac{1}{2}|\alpha|^2)\exp(\alpha a^+)\exp(-\alpha^* a) \tag{12}$$

we obtain

$$\gamma_n(c) = \int_c (\mathrm{d}X_1\, X_2 - \mathrm{d}X_2\, X_1) \tag{13}$$

which, on using the Green theorem in a plane, becomes

$$\gamma_n(c) = -2 \times (\text{area of the circuit}) \tag{14}$$

for all $n$.

*(b) The Hannay angle.* The classical Hamiltonian for this problem is

$$H = \tfrac{1}{2}\{[p-(2\hbar\omega)^{1/2}X_2]^2 + \omega^2[q-(2\hbar/\omega)^{1/2}X_1]^2\}. \tag{15}$$

In terms of action and angle variables $I$ and $w$ the expressions for $p$ and $q$ are (Berry 1985)

$$p = (2\hbar\omega)^{1/2}X_2 - (2I\omega)^{1/2}\sin w \tag{16a}$$

$$q = (2\hbar/\omega)^{1/2}X_1 + (2I/\omega)^{1/2}\cos w. \tag{16b}$$

Substituting from (16) in (4) we get

$$\Delta w(I, c) = 0 \tag{17}$$

in agreement with the semiclassical formula (5).

Next we consider spin coherent states.

*(a) Berry's phase.* Consider the Hamiltonian

$$H = X_1 J_1 + X_2 J_2 + X_3 J_3 \tag{18}$$

where $J_i$ are the angular momentum operators and

$$X_1 = B\sin\theta\cos\theta \qquad X_2 = B\sin\theta\sin\theta \qquad X_3 = B\cos\theta. \tag{19}$$

We take $\theta$ and $\phi$ to be slowly varying. This Hamiltonian describes the motion of a spin $J$ in a magnetic field whose magnitude is $B$ and whose direction varies slowly. This Hamiltonian has been investigated by several authors (Berry 1984, Kuratsuji and Iida 1985). The ground state of this Hamiltonian is the spin coherent state (Arrechi *et al* 1973)

$$|\zeta\rangle = \exp[\zeta J_+ - \zeta^* J_-]|J, -J\rangle \tag{20}$$

where $|J, -J\rangle$ is the usual angular momentum eigenstate with $J_z = -J$ and

$$\zeta = \mathrm{e}^{-i\phi}\theta/2. \tag{21}$$

Another useful expression for the spin coherent state (Arrechi *et al* 1973) is

$$|\xi\rangle = (1+|\xi|^2)^{-J}\exp(\xi J_+)|J_+, -J\rangle \tag{22}$$

where $\xi = \mathrm{e}^{-i\theta}\tan(\theta/2)$.

Using (22) it is easily seen that

$$\langle\xi|\mathrm{d}/\mathrm{d}t|\xi\rangle = -\frac{J}{1+|\xi|^2}\left(\xi\frac{\mathrm{d}\xi^*}{\mathrm{d}t} + \xi^*\frac{\mathrm{d}\xi}{\mathrm{d}t}\right) + \frac{\mathrm{d}\xi}{\mathrm{d}t}\langle\xi|J_+|\xi\rangle. \tag{23}$$

Using the results of Arrechi *et al* for the expectation values of an arbitrary product of angular momentum operators between spin coherent states we find

$$\langle\xi|J_+|\xi\rangle = 2J(1+|\xi|^2)^{-1}\xi^*. \tag{24}$$

From (23) and (24) we obtain

$$\langle\xi|\mathrm{d}/\mathrm{d}t|\xi\rangle = -\frac{J}{1+|\xi|^2}\left(\xi\frac{\mathrm{d}\xi^*}{\mathrm{d}t} - \xi^*\frac{\mathrm{d}\xi}{\mathrm{d}t}\right) \tag{25}$$

and hence

$$\gamma(c)=\frac{J}{\mathrm{i}}\int \mathrm{d}t\frac{1}{1+|\xi|^2}\left(\xi\frac{\mathrm{d}\xi^*}{\mathrm{d}t}-\xi^*\frac{\mathrm{d}\xi}{\mathrm{d}t}\right) \tag{26}$$

which can be shown to be $J$ times the solid angle subtended by the circuit at the origin in the parameter space.

We wish to note that the results for the harmonic oscillator coherent states can be obtained as a limiting case of that for spin coherent states. It is known that the Heisenberg algebra of $a$, $a^+$, $a^+a$ and $I$ can be obtained from the SU(2) algebra by a group contraction (Arrechi *et al* 1973). For the coherent states this implies that the harmonic oscillator coherent states can be recovered from the spin coherent states by putting $\zeta=\xi=c\alpha$ and taking the limit $c\to 0, J\to\infty$ such that $2Jc^2=1$. Taking these limits in (26) we get

$$\gamma(c)=\frac{1}{2\mathrm{i}}\int_0^T \mathrm{d}t\left(\alpha\frac{\mathrm{d}\alpha^*}{\mathrm{d}t}-\alpha^*\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right) \tag{27}$$

which is identical to the result (14) given earlier.

*(b) The Hannay angle.* Gozzi and Thacker (1987) have constructed a classical analogue of the Hamiltonian (18) with $J=\frac{1}{2}$ in terms of Grassmann variables and have calculated the Hannay angle for that system and have checked that it is related to the Berry's phase according to the semiclassical formula (5).

Finally, we consider squeezed coherent states.

*(a) Berry's phase.* Consider the Hamiltonian

$$H=D(\alpha)S(\beta)[h\omega(a^+a+\tfrac{1}{2})]S^+(\beta)D^+(\alpha) \tag{28}$$

where $D(\alpha)$ is the same as in (7) and the squeeze operator $S(\beta)$ is given by

$$S(\beta)=\exp(\tfrac{1}{2}\beta a^+a^+-\tfrac{1}{2}\beta^*aa) \tag{29}$$

where $\beta=X_1+\mathrm{i}X_2$.

The Hamiltonian (28) is quadratic in $(a^+-\alpha^*)$ and $(a-\alpha)$ and the full expression for it in terms of $a^+$ and $a$ may be written out using

$$D(\alpha)aD^+(\alpha)=a-\alpha \tag{30}$$

$$S(\beta)aS^+(\beta)=\cosh ra^+-\mathrm{e}^{-\mathrm{i}\theta}\sinh ra \tag{31}$$

with $\beta=r\mathrm{e}^{\mathrm{i}\theta}$. In the following we shall take $\alpha$ to be a constant and the real and imaginary parts $X_1$ and $X_2$ of $\beta$ to be slowly varying with time. This Hamiltonian is thus a particular case of the general quadratic Hamiltonian considered by Berry (1985).

The eigenstates of the Hamiltonian (28) are the so-called squeezed states (Hollenhorst 1980)

$$|n,\alpha,\beta\rangle=D(\alpha)S(\beta)|n\rangle. \tag{32}$$

Berry's phase is given by

$$\gamma_n(c)=\int_c \mathrm{d}X\langle n,\alpha,\beta|\nabla_x|n,\alpha,\beta\rangle$$
$$=\int_c \mathrm{d}X\langle n|S^+(\beta)\nabla_XS(\beta)|n\rangle. \tag{33}$$

The quantity $\langle n|S^{+}(\beta)\nabla_{x}S(\beta)|n\rangle$ can be easily computed from the following expression for $S(\beta)$ (Hollenhorst 1980):

$$S(\beta)=\exp[\tfrac{1}{2}e^{i\theta}(\tanh r)a^{+}a^{+}](\cosh r)^{-1/2}\left(\sum_{n=0}^{\infty}\frac{(\operatorname{sech} r-1)^{n}}{n_{1}}(a^{+})^{n}(a)^{n}\right)$$
$$\times\exp[-\tfrac{1}{2}e^{i\theta}(\tanh r)aa]. \tag{34}$$

If the circuit in the $X_1-X_2$ space is parametrised by $r=r(\theta)$, $0\leqslant\theta\leqslant 2\pi$, then one obtains the following expression for $\gamma_n(c)$:

$$\gamma_n(c)=-(n+\tfrac{1}{2})\int_0^{2\pi}d\theta\sinh^2 r(\theta). \tag{35}$$

For a circle of radius $R$ in $X_1-X_2$ space, (35) gives

$$\gamma_n(c)=-(n+\tfrac{1}{2})2\pi\sinh^2 R. \tag{36}$$

*(b) The Hannay angle.* The classical Hamiltonian $H(p,q,X)$ corresponding to (28) is

$$H=\tfrac{1}{2}[X_1(\omega q)^2+2X_2(\omega q)p+X_3p^2] \tag{37}$$

where

$$X_1=\cosh 2r-\sinh 2r\cos\theta\qquad X_2=\cosh 2r+\sinh 2r\cos\theta\qquad X_3=\sinh 2r.$$

Following Berry (1985) the expressions for $p$ and $q$ in terms of action and angle variables $I$ and $w$ are

$$q=(2X_3I/\omega)^{1/2}\cos w \tag{38}$$

$$p=-(2X_3I/\omega)^{1/2}[(X_2/X_3)\cos w+(\omega/X_3)\sin w]. \tag{39}$$

Substituting this in (4), one finds that the Hannay angle in this case turns out to be

$$\Delta\omega(I,c)=\int_0^{2\pi}d\theta\sinh^2 r(\theta) \tag{40}$$

as would be expected from the semiclassical formula (5).

To conclude, we hope that our results will be useful in verifying the existence of Berry's phase through quantum optics experiments.

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